Chain Conditions On Rings And Modules

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Brief Outline

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Modules

- A module is basically an abelian group on which a ring acts.

  Definition:

  Let R be a ring. A left R-module of R is an abelian group M together with a action $(r, x) \longmapsto rx$ of R on M such that

  1. $r(sx) = (rs)x$
  2. $(r + s)x = rx + sx, r(x + y) = rx + ry$

     for all $r, s \in R$ and $x, y \in M$.

     If R has an identity element, then a left R-module M is unital when
  3. $1x = x$ for all $x \in M$.

Modules

- Example:
    1. A vector space over a field K is (exactly) a unital left K-module.
    2. Every abelian group A is a unital $\mathbb{Z}$-module, in which nx is the usual integer multiple, $nx = x + x + x + ...x$ when $n \in \mathbb{N}$, $x \in A$.
    3. Every ring R acts on itself by left multiplication. This makes R as a left R-module, denoted by R^R to distinguish it from the ring R. If the ring has an identity then R^R is unital.

Sub Module

- Definition: A submodule of a left R-module M is an additive subgroup A of M such that $x \in A$ implies $rx \in A$ for all $r \in R$.
- Example:
  1. $\{0\}$ and M are submodules of any R-module of M.
  2. Submodules of a vector space are its subspaces.
  3. Submodules of an abelian group (as $\mathbb{Z}$-module) are its subgroups.

Quotient Module

- Definition: Let M be a R-module and M' be a submodule of M. Then M/M' is an abelian group inherits a R-module structure from M, defined by $r(x + M') = rx + M'$. M/M' is called quotient M modulo M'.
- Example:
  1. Let G be an abelian group and G' be a subgroup of G. Then G/G' is a quotient module (as $\mathbb{Z}$-module).
  2. Let V be a vector space over a field F and W is a subspace of it. Then V/W is a quotient module (as F-module).
  3. Let R be a commutative ring and R' be an ideal of R. Then R/R' is a quotient module (as R-module).

Homomorphism

- Definition: Let A and B be left R-modules. A homomorphism $\varphi : A \longmapsto B$ of left R-modules is a mapping $\varphi : A \longmapsto B$ such that
  1. $\varphi(x + y) = \varphi(x) + \varphi(y)$.
  2. $\varphi(rx) = r\varphi(x)$

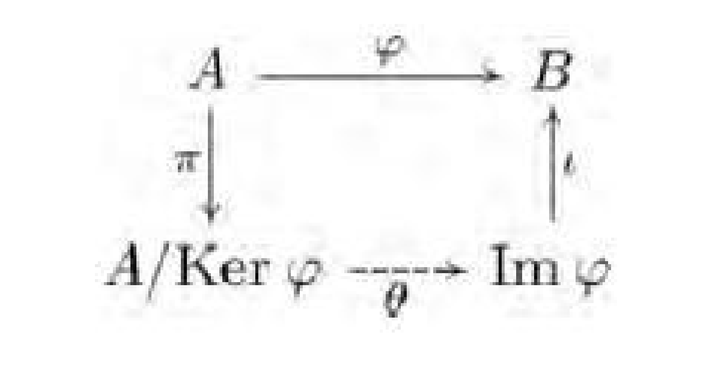
  $\forall x, y \in A$ and $r \in R$.
- Example:
  1. Let $\varphi : \mathbb{Z}/4\mathbb{Z} \longmapsto \mathbb{Z}/2\mathbb{Z}$ defined as $\varphi(x + 4\mathbb{Z}) = x + 2\mathbb{Z}$ is a module homomorphism where $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ are $\mathbb{Z}$-modules.
  2. Let V_1 and V_2 be two vector spaces over some field say F. Then any linear transformation φ from V_1 to V_2 is a module homomorphism.
- Definition: Let $\varphi : A \longmapsto B$ be a module homomorphism. The image or range of φ is $Im\phi = \{\varphi(x) : x \in A\}$. The kernel of φ is defined as $Ker\varphi = \{x \in A : \varphi(x) = 0\} = \varphi^{-1}(0)$.

Isomorphism

- Definition: An one-one onto module homomorphism is a module isomorphism.
- Homomorphism Theorem: If $\varphi : A \longmapsto B$ is a homomorphisms of left R-modules, then

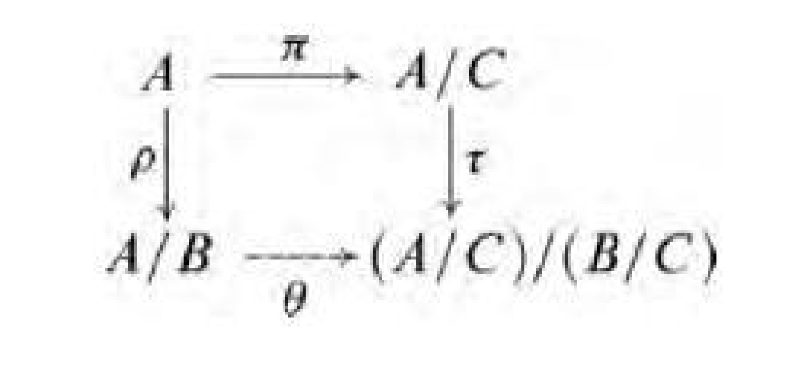
$$A/Ker\varphi \cong Im\varphi;$$

in fact there is an isomorphism $\theta : A/Ker\varphi \longmapsto Im\varphi$ unique such that $\varphi = \iota o\theta o\pi$, where $\iota : Im\varphi \longmapsto B$ is an inclusion homomorphism and $\pi : A \longmapsto A/Ker\varphi$ is the canonical projection.



Isomorphism

- First Isomorphism Theorem: If A is a left R-module and $B \supseteq C$ are submodules of A, then
$$A/B \cong (A/C)/(B/C);$$
in fact there is a unique isomorphism $\theta : A/B \longmapsto (A/C)/(B/C)$ such that $\theta o \rho = \tau o \pi$, where $\pi : A \longmapsto A/C$, $\rho : A \longmapsto A/B$, and $\tau : A/C \longmapsto (A/C)/(B/C)$ are canonical projections.

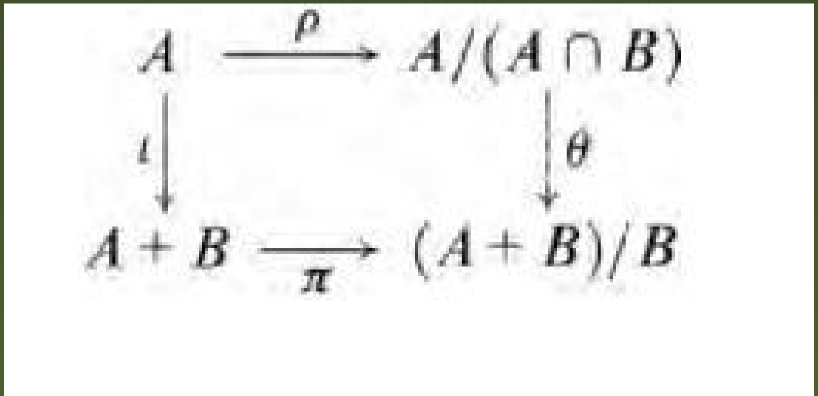


Isomorphism

- Second Isomorphism Theorem: If A and B are two submodules of a left R-module, then

$$(A+B)/B \cong A/(A\cap B);$$

in fact, there is an isomorphism $\theta : A/(A\cap B) \longmapsto (A+B)/B$ unique such that $\theta o \rho = \pi o \iota$, where $\pi : A+B \longmapsto (A+B)/B$ and $\rho A \longmapsto A/(A\cap B)$ are the canonical projections and $\iota : A \longmapsto A+B$ is the inclusion homomorphism.



Direct Sum And Direct Product

- Definition: Let $(M_i)_{i\in I}$ is any family of left R-modules where I is some index set. Then $\bigoplus_{i\in I} M_i = \{(x_i)_{i\in I} : x_i \in M_i, i \in I$ and $x_i \neq 0$ for finitely many $i \in I\}$ is called the direct sum of the modules M_i.
- Example:
- $C_{00} = \bigoplus_{i\in\mathbb{N}} M_i$ where $M_i =$span$\{(0, 0, 0, ..., 1, 0, 0, ..)\}$ over $\mathbb{R}$.
- Definition: Let $(M_i)_{i\in I}$ is any family of left R-modules where I is some index set. Then $\Pi_{i\in I} M_i = \{(x_i)_{i\in I} : x_i \in M_i, i \in I\}$ is called the direct product of the modules M_i.
- Example: $\mathbb{R}^I = \Pi_{\alpha\in I}\mathbb{R}$ where I is an index set.

Exact Sequence

- Definition: A sequence of left R-modules and R-homomorphisms
  $$\ldots.. \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \ldots..$$
  is said to be exact at M_i if $Im(f_i) = Ker(f_{i+1})$. The sequence is exact if it i
  exact at each M_i.
- Theorem: Let M, M', M'' are three left/right R-modules.Then we have the followings:
  1. $0 \longrightarrow M' \xrightarrow{f} M$ is exact $\Leftrightarrow$ f is injective.
  2. $M \xrightarrow{g} M'' \longrightarrow 0$ is exact $\Leftrightarrow$ g is surjective.
  3. $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is exact $\Leftrightarrow$ f is injective and g is surjective and g induces an isomorphism of $Coker(f) = M/f(M')$ onto M''.

Chain Conditions

- Definition: Let $\sum$ be a set with a partial order relation $\leq$. Then we have the following
  The following conditions on $\sum$ are equivalent
  1. Every increasing sequence $x_1 \leq x_2 \leq x_3 \leq ..... \leq x_n \leq x_{n+1} \leq ....$ in $\sum$ is stationary in $\sum$.
  2. Every non empty subset of $\sum$ has a maximal element.
- If $\sum$ is the set of submodules of a left R-module M, with the partial order relation be $\subseteq$ then the above is called the ascending chain condition.
- Moreover we can consider decreasing sequence of sub modules of a left R-module which is stationary, then it is called descending chain condition.

Chain Conditions

- *Noetherian* Module:
  A left R-module is called a left *Noetherian* module if it satisfies the ascending chain condition(a.c.c in short).
  The name *Noetherian* is given after the name of the Mathematician *Emmy Noether*.
- *Artinian* Module:
  A left R-module is called left *Artinian* module if it satisfies the descending chain condition(d.c.c in short).
  The name *Artinian* was given after the name of the Mathematician *Emil Artin*.

Noetherian And *Artinian* Modules

- Let A be a finite abelian group (as a $\mathbb{Z}$-module) satisfies both a.c.c. and d.c.c. hence *Noetherian* and *Artinian* module.
- The ring $\mathbb{Z}$ (as a $\mathbb{Z}$-module) satisfies a.c.c. but not d.c.c.
  For if $a \in \mathbb{Z}$ and $a \neq 0$ then we have
  $(a) \supset (a^2) \supset (a^3) \supset (a^4) \supset .... \supset (a^n) \supset ..$
  which is a strict inclusion and hence *Noetherian* but not *Artinian*.
- Let G be a subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of elements whose order is a power of p for some fixed prime p.
  Then G has only one subgroup G_n of order p^n for each $n \geq 0$,
  and $G_0 \subset G_1 \subset G_2 \subset G_3 \subset ..... \subset G_n \subset ......,$
  strict inclusion hence does not satisfy the a.c.c.
  But, only subgroups of G are G_ns and hence satisfies d.c.c. hence not *Noetherian* but *Artinian*.

Noetherian And *Artinian* Modules

- Let us consider $K[x_1, x_2, x_3, ...]$ where K is a field. Then, this does not satisfy a.c.c. and d.c.c. As we have
  $(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset ..... \subset (x_1, x_2, x_3, ..., x_n) \subset .....$ and
  $(x_1) \supset (x_1^2) \supset (x_1^3) \supset ..... \supset (x_1^n) \supset ....$
  Hence $K[x_1, x_2, x_3, ...]$ is neither *Noetherian* not *Artinian*.

Propreties Of *Noetherian* And *Artinian* Modules

- Theorem: Let M be a left *Noetherian* R-module if and only if every sub modules of M are finitely generated.
- Theorem: Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence. Then M is left *Noetherian*/*Artinian* R-module if and only if M' and M'' are left *Noetherian*/*Artinian* R-module.
- Corollary: Let N be a submodule of a left R-module M. Then M/N and N are left *Noetherian*/*Artinian* R-module if and only if M is left *Noetherian*/*Artinian* R-module.

Propreties Of *Noetherian* And *Artinian* Modules

- Theorem: If M_i are left *Noetherian*/*Artinian* R-modules where $i \leq i \leq n$ then $\bigoplus_{i=1}^{n} M_i$ is also left *Noetherian*/*Artinian* R-module.
- Theorem: Let φ is a surjective module homomorphism from a left *Noetherian* module M to itself. Then φ is an isomorphism.
- Theorem: Let φ is a injective module homomorphism from a left *Artinian* module M to itself. Then φ is an isomorphism.

Composition Series

- Definition: Let a chain of submodules of a module M

$$M = M_0 \supset M_1 \supset M_2 \supset .... \supset M_n = 0$$

is called a composition series of M (length n or n links) if M_{i-1}/M_i for all $1 \leq i \leq n$ is simple.

- Example: Take $M = \mathbb{Z}/16\mathbb{Z}$ as as $\mathbb{Z}$-module. Then, $M =< 1 >\supset< 2 >\supset< 4 >\supset< 8 >\supset 0$ is a composition series.
- Theorem: Suppose M has a composition series of finite length n. Then every composition series of M has length n, and every chain in M can be extended to a composition series.
- Theorem: Suppose M has a composition series of finite length n. Then every composition series of M has length n, and every chain in M can be extended to a composition series.
- Theorem: M be a left R-module, has a composition series if and only if M satisfies both the chain conditions.

Composition Series

- Definition: A left module satisfying both a.c.c. and d.c.c is called the module of finite length.
- Theorem: For a vector space V over a field F the following conditions are equivalent:
  1. V is finite dimensional;
  2. V is of finite length;
  3. V satisfies a.c.c.
  4. V satisfies d.c.c.

Noetherian Rings

- Definition: A ring R is said to be left *Noetherian* if satisfies one of the equivalent property
  1. Every non-empty set of left ideals in R has a maximal element.
  2. Every ascending chain of left ideals in R is stationary.
  3. Every left ideal is finitely generated.
- Example:
  1. The ring $\mathbb{Z}$ is *Noetherian*.
  2. Every field is *Noetherian*.
  3. Any finite ring is *Noetherian*.
  4. Every principal ideal domain is *Noetherian*.

Properties Of *Noetherian* Rings

- Theorem: Let R be a *Noetherian* ring and a is an ideal of it. Then R/a is also a *Noetherian* ring.
- Theorem: If R be a *Noetherian* ring and φ is a homomorphism from R onto ring R_1.
  Then R_1 is also *Noetherian*.
- Theorem: Let R be a *Noetherian* ring, and M is a finitely generated R-module. Then M *Noetherian*.
- Corollary: Let R_1 be a subring of R. Suppose R_1 is *Noetherian* ring and R is finitely generated R_1 module, then R is a *Noetherian* ring.
- Example: $\mathbb{Z}[i]$ is Noetherian ring as $\mathbb{Z}$ is Noetherian ring and a subring of $\mathbb{Z}[i]$
- Theorem: Matrix over *Noetherian* ring with unity is *Noetherian*.
- Example: $M_n(\mathbb{Z})$, $M_n(\mathbb{Q})$, $M_n(\mathbb{Z}_2)$ are a Noetherian rings.

Hilbert Basis Theorem

- The original statement of the Hilbert Basis theorem was as follows:
  Every ideal of $\mathbb{C}[X_1, X_2, ..., X_n]$ has a finite basis.
  We shall state here the more general version of the Hilbert Basis Theorem.
- *Hilbert* Basis Theorem: R be a commutative ring with identity 1. If R is a *Noetherian* then $R[X]$ is *Noetherian* ring.
- Corollary: If R is a commutative *Noetherian* ring with identity then $R[X_1, X_2, X_3, ...X_n]$ is also *Noetherian* ring.

Artinian Rings

- Definition: A ring R is said to be left *Artinian* if satisfies one of the equivalent property
  1. Every non-empty set of left ideals in R has a minimal element.
  2. Every ascending chain of left ideals in R is stationary.
- Example:
  1. Every field is *Artinian*.
  2. Any finite ring is *Artinian*.

Properties Of *Artinian* Rings

- Theorem: Let R be a *Artinian* ring and a is an ideal of it.
  Then R/a is also a *Artinian* ring.
- Theorem: If R be a *Artinian* ring and φ is a homomorphism from R onto a ring R_1.
  Then R_1 is also *Artinian*.
- Theorem: Let R be a *Artinian* ring, and M is a finitely generated R-module.
  Then M *Artinian*.
- Corollary: Let R_1 be a subring of R. Suppose R_1 is *Artinian* ring and R is finitely generated R_1 module, then R is a *Artinian* ring.
- Theorem: Matrix over *Artinian* ring with unity is *Artinian*.
- Example: $M_n(\mathbb{Q})$, $M_n(\mathbb{Z}_2)$ are a Artinian rings.

Properties Of *Artinian* Rings

- Theorem: In an *Artinian* ring R every prime ideal is maximal.
- Definition: In a ring R intersection of all prime ideals of R is an ideal and is called the nil radical.
- Definition: In a ring R intersection of all maximal ideals is an ideal and it is called the jacobson radical.
- Corollary: In an *Artinian* ring every nil radical is Jacobson radical.
- Theorem: An *Artinian* ring has only finite number of maximal ideals.

Properties Of *Noetherian* Space

- Definition: A topological space X is said to be *Noetherian* space if every chain of open subsets of X satisfies a.c.c. or equivalently every every chain of closed subsets of X satisfies d.c.c.
- Example:
  1. Let X is a finite set with any topology is a *Noetherian* space.
  2. $\mathbb{R}$ with finite complement topology is a *Noetherian* Space.
- Theorem: If X is a *Noetherian* space if and only if every subspace of X is quasicompact.

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Future Plan for this work

- Further studies of the *Noetherian* and *Artinian* rings
- *Dedekind* Domains
- Semisimple rings and localization
- *Wedderburn's* theorems
- Introduction to the dimension theory
- We would like to study one particular *Noetherian* local ring $\Lambda = \mathbb{Z}_p[[T]]$, because of its importance in aritmetic geometry.

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